Positive solutions to some cross-diffusion systems in population dynamics (Dynamics of spatio - temporal patterns for the system of reaction - diffusion equations)

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Positive solutions to some cross-diffusion systems in population dynamics

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1 Introduction

In this article, we are concerned with the following strongly coupled parabolic system;

$$
\text{(P)}\begin{cases}
u_t = \Delta[(1+\alpha v)u] + u(a-u-cv) & \text{in } \Omega\times(0,T),\\
v_t = \Delta\left[\left(\mu + \dfrac{1}{1+\beta u}\right)v\right] + v(b+du-v) & \text{in } \Omega\times(0,T),\\
u = v = 0 & \text{on } \partial\Omega\times(0,T),\\
u(\,\cdot\,,t) = u_0 \geq 0,\ \ v(\,\cdot\,,t) = v_0 \geq 0 & \text{on } \Omega,
\end{cases}
$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ with a smooth boundary $\partial\Omega$; a, b, c, d and μ are all positive constants; α and β are nonnegative constants. System (P) is one of Lotka-Volterra prey-predator models with nonlinear diffusion effects. From such an ecological model point of view, unknown functions u and v represent population densities of prey and predator, respectively. In reaction terms, a and b are birth rates of respective species, c and d mean prey-predator interactions. In the first equation, the nonlinear diffusion term $\alpha\Delta(uv)$ describes a tendency such that the prey species keep away from high density areas of the predator species. This term $\alpha\Delta(uv)$ is usually referred as the *cross-diffusion* term. A competition population model with cross-diffusion terms was first proposed by Shigesada-Kawasaki-Teramoto [29]. Since their pioneer work, many mathematicians have discussed population models with cross-diffusion terms from various view points, e.g., the global existence of time-depending solutions ([1], [3], [8], [9], [10], [24], [30]) and steady-state problems ([13], [14], [16], [21], [22], [23], [25], [26], [28]). In the second equation, the fractional type nonlinear diffusion $\Delta(\frac{v}{1+\beta u})$ models a situation such that the population pressure of the predator species weakens in high

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density areas of the prey species. To my knowledge, there are few works about such fractional type nonlinear diffusion effects in a field of reaction-diffusion systems.

In the present article, we will mainly discuss the associate steady state problem;

$$
\text{(SP)}\begin{cases}\Delta[(1+\alpha v)u]+u(a-u-cv)=0 & \text{in } \Omega,\\ \Delta\left[\left(\mu+\dfrac{1}{1+\beta u}\right)v\right]+v(b+du-v)=0 & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega.\end{cases}
$$

Among other things, we are interested in positive solutions of (SP). From the view point of the prey-predator model, a positive solution (u, v) means a *coexistence* steady state. So it is important to study the positive solution set of (SP). Our first aim is to obtain a sufficient condition of coefficients $(\alpha, \beta, \mu, a, b, c, d)$ for existence of positive solutions to (SP). Our approach to the proof is based on the bifurcation arguments. Throughout the article, we will regard the coefficient a as a positive bifurcation parameter. Our strategy is to seek a bifurcation point on the semitrivial solution sets by making use of the local bifurcation theory ([4]). Here a semitrivial solution means a solution (u, v) such that either u or v vanishes in Ω. We will find a certain number $a^* = a^*(\alpha, \mu, b, c, d)$ such that positive solutions bifurcate from the semitrivial solution with $u \equiv 0$ at $a = a^*$, if $b > (\mu+1)\lambda_1$, where λ_1 is denoted by the least eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$. On the other hand, if $b < (\mu+1)\lambda_1$, we will get a certain $a_* = a_*(\beta, \mu, b, c, d)$ such that positive solutions bifurcate from the semitrivial solution with $v \equiv 0$ at $a = a_*$. By a combination with the global bifurcation theory ([27]) and some apriori estimates for positive solutions, we will prove that the positive solution branch bifurcates from a semitrivial solution at $a = a^*$ or $a = a_*$ and extends globally with respect to a. Therefore, we know that (SP) admits at least one positive solution if $b > (\mu+1)\lambda_1$ (resp. $b < (\mu+1)\lambda_1$) and $a > a^*$ (resp. $a > a_*$).

Our second aim is to derive a large nonlinear diffusion effect of β on the positive solution set to (SP) with a case when $\alpha = 0$ and $b > (\mu+1)\lambda_1$. For the sake of this derivation, we will introduce two *shadow systems* as $\beta \to \infty$ in (SP) with $\alpha = 0$. Let $\{\beta_n\}$ be any sequence with $\lim_{n\to\infty}\beta_n = \infty$ and suppose that $\{(u_n, v_n)\}$ is any positive solution sequence to (SP) with $\alpha = 0$ and $\beta = \beta_n$. Under some additional assumptions, we will prove that subject to a subsequence, one of the following two cases necessarily occurs:

(i) There exists a certain positive solution (u, v) of

$$
\begin{cases}\Delta u+u(a-u-cv)=0 & \text{in } \Omega,\\ \mu\Delta v+v(b+du-v)=0 & \text{in } \Omega,\\ u=v=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}
$$

such that $\lim_{n\to\infty}(u_n, v_n) = (u, v)$ in $L^\infty(\Omega)^2$.

(ii) There exists a certain positive solution (w, v) of

$$\begin{cases} \Delta w + w(a - cv) = 0 & \text{in } \Omega, \\ \Delta\left[\left(\mu + \dfrac{1}{1+w}\right)v\right] + v(b - v) = 0 & \text{in } \Omega, \\ w = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

such that $\lim_{n\to\infty}(\beta_n u_n, v_n) = (w, v)$ in $L^\infty(\Omega)^2$.

Our convergence result (Theorem 3.1) will also assert that if β is sufficiently large, any positive solution of (SP) (with $\alpha = 0$) can be approximated by a certain positive solution of either (1.1) or (1.2). So it is natural to ask which of (1.1) or (1.2) (or both) can characterize positive solutions of (SP), in each coefficient (μ, a, b, c, d) case. There are many studies about the first shadow system (1.1) (see e.g., [2], [5], [6], [7], [17], [18], [19], [20], [31]). According to their results, for any (μ, b, c, d) fixed, we have a threshold number $\hat{a}$ $(> a^*)$ such that (1.1) admits a positive solution if and only if $a > \hat{a}$. Thus it is a crucial part of this article to study the positive solution set of the second shadow system (1.2). By regarding a as a bifurcation parameter, we will show that the branch of the positive solution set of (1.2) bifurcates from a semitrivial solution with $w \equiv 0$ at $a = a^*$, and extends globally with respect to w. (The branch is uniformly bounded with respect to (v, a).) Furthermore, we will prove that the branch necessarily blows up with respect to $\|w\|_\infty$ at $a = \hat{a}$. So this result also implies that positive solution set of (SP) (with $\alpha = 0$) structurally changes near $a = \hat{a}$, when β is sufficiently large (Theorem 3.8).

Throughout the article, we will denote by $\lambda_1(q)$ the least eigenvalue of the problem

$$-\Delta u + q(x)u = \lambda u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where $q(x)$ is a continuous function in $\overline{\Omega}$. We simply write λ_1 instead of $\lambda_1(0)$. It is well known that the following problem

$$\Delta u + u(a - u) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \tag{1.3}$$

has a unique positive solution $u = \theta_a$ if and only if $a > \lambda_1$. Then (SP) has a semitrivial solution $(u, v) = (\theta_a, 0)$ if $a > \lambda_1$. Furthermore it is easily verified that (SP) has another semitrivial solution $(u, v) = (0, (\mu + 1)\theta_{b/(\mu+1)})$ if $b > (\mu + 1)\lambda_1$. Here, $\theta_{b/(\mu+1)}$ represents a positive solution of (1.3) with a replaced by $b/(\mu + 1)$. The usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|u\|_p := \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x\in\overline{\Omega}} |u(x)|.$$

In particular, we simply write $\|u\|$ instead of $\|u\|_2$.

The contents of the present article are as follows: In Section 2, we first give the sufficient condition for existence of positive solutions to (SP). Next we give the outline of the proof. In Section 3, we will discuss a special case when β is sufficiently large. The above convergence to one of two shadow systems as $\beta \to \infty$ will be justified in this section. The solution set of (1.2) will be studied in the latter half of this section.

2 Coexistence Region

2.1 Main Result

In this section, we first give a sufficient condition of existence of positive solutions to (SP).

Theorem 2.1. *If $a \leq \lambda_1$, then* (SP) *has no positive solution. In a case when $a > \lambda_1$,* (SP) *admits a positive solution if the following condition* (2.1) *holds true.*

$$\lambda_1\left(\frac{c(\mu+1)\theta_{b/(\mu+1)}-a}{1+\alpha(\mu+1)\theta_{b/(\mu+1)}}\right)<0 \ \ and \ \ \lambda_1\left(-\frac{(b+d\theta_a)(1+\beta\theta_a)}{\mu(1+\beta\theta_a)+1}\right)<0. \tag{2.1}$$

Here it is defined that $\theta_{b/(\mu+1)} \equiv 0$ if $b \leq (\mu+1)\lambda_1$.

We need to explain the meaning of Theorem 2.1. Regarding a and b as positive parameters, we introduce the following two sets in the (a, b) plane,

$$S_1 := \left\{(a,b)\in \boldsymbol{R}_+^2 \ : \ \lambda_1\left(-\frac{(b+d\theta_a)(1+\beta\theta_a)}{\mu(1+\beta\theta_a)+1}\right)=0 \ \text{ for } \ a\geq\lambda_1\right\},$$
$$S_2 := \left\{(a,b)\in \boldsymbol{R}_+^2 \ : \ \lambda_1\left(\frac{c(\mu+1)\theta_{b/(\mu+1)}-a}{1+\alpha(\mu+1)\theta_{b/(\mu+1)}}\right)=0 \ \text{ for } \ b\geq(\mu+1)\lambda_1\right\}.$$

The following Lemmas 2.2 and 2.3 mention the shapes of curves S_1 and S_2, respectively. See [12] for the proofs of Lemmas 2.2 and 2.3.

Lemma 2.2. *There exists a certain $a_0 > \lambda_1$ such that S_1 can be expressed as*

$$S_1 = \{(a,b)\in \boldsymbol{R}_+^2 \ : \ b=\underline{b}(a) \ \textit{for} \ \lambda_1\leq a<a_0\},$$

where $b = \underline{b}(a)$ is a positive continuous function for $a \in [\lambda_1, a_0)$, and satisfies the following properties :

(i) $\underline{b}(\lambda_1) = (\mu+1)\lambda_1, \ \lim_{a\to a_0}\underline{b}(a) = 0.$

(ii) *$b = \underline{b}(a)$ is monotone decreasing with respect to $a \in [\lambda_1, a_0]$.*

Lemma 2.3. *The set S_2 possesses the expression*

$$S_2 = \{(a,b) \in \boldsymbol{R}_+^2 \; : \; b = \overline{b}(a) \; \text{for} \; a \geq \lambda_1\},$$

where $b = \overline{b}(a)$ is a positive continuous function for $a \in [\lambda_1, \infty)$, and satisfies the following properties :

(i) *$\overline{b}(a)$ is a monotone increasing function with respect to a.*

(ii) $\overline{b}(\lambda_1) = (\mu + 1)\lambda_1, \; \lim_{a \to \infty} \overline{b}(a) = \infty.$

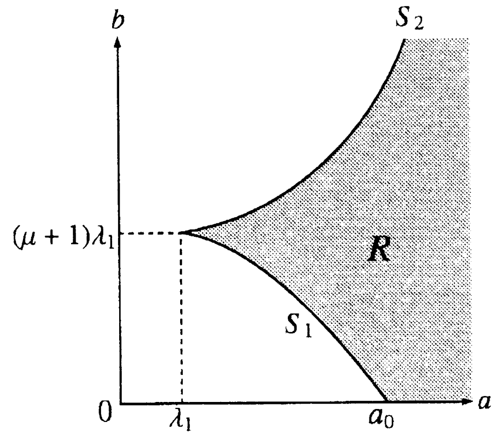

Fig.1 : Coexistence Region

Combining these properties of S_1 and S_2, one can deduce from Theorem 2.1 that if (a,b) lies in a region R surrounded by S_1 and S_2, then (SP) has a positive solution (see Fig. 1). This R, in case $\alpha = \beta = 0$, corresponds to the exact coexistence region shown by López-Gómez and Pardo [19]. From a view-point of the bifurcation theory, we will prove that positive solutions bifurcate from $(u,v) = (\theta_a, 0)$ when (a,b) crosses S_1 curve. Similarly positive solutions also bifurcate from $(u,v) = (0, (\mu + 1)\theta_{b/(\mu+1)})$ when (a,b) moves across S_2.

2.2 Apriori Estimates

In the rest part of the section, we give the outline of the proof of Theorem 2.1. In this subsection, we first introduce a semilinear elliptic system equivalent to (SP), and next give some apriori estimates of positive solutions to the semilinear system. Such apriori estimates will make an important rules in the proof. Assume $(\alpha, \beta) \neq (0,0)$ in (SP). As long as we are restricted on nonnegative solutions, it is convenient to introduce two unknown functions U and V by

$$U = (1 + \alpha v)u \quad \text{and} \quad V = \left(\mu + \frac{1}{1 + \beta u}\right) v \tag{2.2}$$

There is a one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$. It is possible to describe their relations by

$$
\begin{aligned}
& u = u(U, V) \\
& = \frac{-(\alpha V + 1) + \mu(\beta - 1) + \sqrt{\{(\alpha V + 1) - \mu(\beta - 1)\}^2 + 4\beta(\alpha V + \mu)(1 + \mu)}}{2\beta(\alpha V + \mu)}, \\
& v = v(U, V) \\
& = \frac{\alpha V - 1 - \mu(\beta U + 1) + \sqrt{\{(\alpha V - 1) - \mu(\beta U + 1)\}^2 + 4\alpha V(\beta U + 1)(1 + \mu)}}{2\alpha(1 + \mu)}.
\end{aligned}
\tag{2.3}
$$

Since we are concerned with nonnegative solutions, (SP) is rewritten in the following equivalent form

$$
\text{(EP)} \begin{cases} \Delta U + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta V + v(b + du - v) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood as functions of (U, V) defined by (2.3). It is easy to show that (EP) has two semitrivial solutions

$$
(U, V) = (\theta_a, 0) \text{ for } a > \lambda_1 \quad \text{and} \quad (U, V) = (0, (\mu + 1)^2 \theta_{b/(\mu+1)}) \text{ for } b > (\mu + 1)\lambda_1,
$$

in addition to the trivial solution $(U, V) = (0, 0)$. We obtain the following apriori estimates for positive solutions of (EP).

Lemma 2.4. *Suppose that (U, V) is any positive solution of* (EP) *and that (u, v) is any positive solution of* (SP). *Then, for all $x \in \Omega$,*

$$
0 \leq u(x) \leq U(x) < M = M(a) := \begin{cases} a & \text{if } \alpha a \leq c, \\ \dfrac{(c + \alpha a)^2}{4\alpha c} & \text{if } \alpha a \geq c, \end{cases}
\tag{2.4}
$$

$$
0 \leq V(x) < \begin{cases} \left(\mu + \dfrac{1}{1 + \beta M}\right)(b + dM) & \text{if } b\beta \leq d, \\ \mu(b + dM) + b & \text{if } b\beta > d, \end{cases}
$$

$$
0 \leq V(x) \leq v(x) < \begin{cases} \dfrac{1}{\mu}\left(\mu + \dfrac{1}{1 + \beta M}\right)(b + dM) & \text{if } b\beta \leq d, \\ (b + dM) + \dfrac{b}{\mu} & \text{if } b\beta > d. \end{cases}
$$

We refer [12] for the proof of Lemma 2.4. The next lemma yields a lower bound for $V(x)$ in a special case when $b > (\mu + 1)\lambda_1$.

Lemma 2.5. *Let* (U, V) *be any positive solution of* (EP). *If* $b > (\mu + 1)\lambda_1$, *then*

$$V(x) \geq \mu^2 \theta_{b/(\mu+1)}(x) \text{ for all } x \in \overline{\Omega}.$$

Proof. It follows from the second equation of (EP) and (2.2) that

$$-\Delta V = v(b + du - v) > v(b - v) = \frac{V}{\mu + \frac{1}{1+\beta u}} \left(b - \frac{V}{\mu + \frac{1}{1+\beta u}} \right).$$

Therefore, we obtain

$$-\Delta V > V\left(\frac{b}{\mu + 1} - \frac{V}{\mu^2}\right) \text{ in } \Omega.$$

By the well known comparison theorem, we immediately obtain the assertion. Then the proof of Lemma 2.5 is accomplished.

□

The following lemma gives a nonexistence region for positive solutions of (EP).

Lemma 2.6. *If* $a \leq \lambda_1$ *or* $(1 + \beta M(a))(b + dM(a)) \leq \lambda_1$, *then* (EP) (*or equivalently,* (SP)) *has no positive solution. Here* $M(a)$ *is the positive number defined in* (2.4).

Proof. Suppose for contradiction that (U, V) is a positive solution of (EP) with the case $(1 + \beta M(a))(b + dM(a)) \leq \lambda_1$. Since $u \leq U \leq M(a)$ by Lemma 2.4, then

$$-\Delta V = v(b + du - v) = V(1 + \beta u)(b + du - v) < (1 + \beta M(a))(b + dM(a))V$$

in Ω. Then by taking $L^2(\Omega)$ inner product with V, we obtain

$$||\nabla V||^2 < (1 + \beta M(a))(b + dM(a))||V||^2. \tag{2.5}$$

Since $||\nabla V||^2 \geq \lambda_1 ||V||^2$ by Poincaré's inequality, (2.5) obviously yields a contradiction. By virtue of $U(a - u - cv)/(1 + \alpha v) < aU$ in Ω, one can derive the assertion for the case $a \leq \lambda_1$ in a similar manner.

□

2.3 Bifurcations from Semitrivial Solutions

In this subsection, we will find bifurcation points on the semitrivial solution sets of (EP) with regarding a as a parameter. Let a be a bifurcation parameter and assume that all other constants are fixed. Concerning (EP), we will obtain a positive solution branch which bifurcates from the semitrivial solution curve

$$\{(U, V, a) : (U, V) = (\theta_a, 0),\ a > \lambda_1\} \text{ or}$$
$$\{(U, V, a) : (U, V) = (0, (\mu + 1)^2 \theta_{b/(\mu+1)}),\ a > \lambda_1\}.$$

By virtue of Lemma 2.2, if $b < (\mu + 1)\lambda_1$, then there exists a unique $a_* \in (\lambda_1, \infty)$ such that

$$\lambda_1 \left(-\frac{(b + d\theta_{a_*})(1 + \beta\theta_{a_*})}{\mu(1 + \beta\theta_a) + 1} \right) = 0. \tag{2.6}$$

On the other hand, if $b > (\mu + 1)\lambda_1$, Lemma 2.3 yields a unique $a^* \in (\lambda_1, \infty)$ such that

$$\lambda_1 \left(\frac{c(\mu + 1)\theta_{b/(\mu+1)} - a^*}{1 + \alpha(\mu + 1)\theta_{b/(\mu+1)}} \right) = 0. \tag{2.7}$$

In view of (2.6) and (2.7), we introduce two positive functions ϕ_* and ϕ^* by solutions to the problems

$$-\Delta\phi_* - \frac{(b + d\theta_{a_*})(1 + \beta\theta_{a_*})}{\mu(1 + \beta\theta_a) + 1}\phi_* = 0 \ \text{ in } \Omega, \quad \phi_* = 0 \ \text{ on } \partial\Omega, \quad \|\phi_*\| = 1$$

and

$$-\Delta\phi^* + \frac{c(\mu + 1)\theta_{b/(\mu+1)} - a^*}{1 + \alpha(\mu + 1)\theta_{b/(\mu+1)}}\phi^* = 0 \ \text{ in } \Omega, \quad \phi^* = 0 \ \text{ on } \partial\Omega, \quad \|\phi^*\| = 1, \tag{2.8}$$

respectively. For $p > N$, we define Banach spaces X and Y by

$$\begin{cases} X := [\, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,] \times [\, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,], \\ Y := L^p(\Omega) \times L^p(\Omega). \end{cases}$$

Lemma 2.7. *Suppose that $a > \lambda_1$. Then the following local bifurcation properties* (i) *and* (ii) *hold true*:

(i) *Let $b < (\mu + 1)\lambda_1$. Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve $\{(\theta_a, 0, a) : a > \lambda_1\}$ if and only if $a = a_*$. To be precise, all positive solutions of* (EP) *near $(\theta_{a_*}, 0, a_*) \in X \times \boldsymbol{R}$ can be expressed as*

$$\Gamma_* = \{(\theta_{a_*} + s(\psi + \hat{U}(s)),\ s(\phi_* + \hat{V}(s)),\ a(s)) \ : \ 0 < s \le \delta\}$$

for some $\psi \in X$ and $\delta > 0$. Here $(\hat{U}(s), \hat{V}(s), a(s))$ is a smooth function with respect to s and satisfies $(\hat{U}(0), \hat{V}(0), a(0)) = (0, 0, a_*)$ and $\int_\Omega \hat{V}(s)\phi_* = 0$.

(ii) *Let $b > (\mu + 1)\lambda_1$. Then positive solutions of* (EP) *bifurcate from the semitrivial solution curve $\{(0, (\mu+1)^2\theta_{b/(\mu+1)}, a) : a > \lambda_1\}$ if and only if $a = a^*$. More precisely, all positive solutions of* (EP) *near $(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a^*) \in X \times \boldsymbol{R}$ are given by*

$$\Gamma^* = \{(s(\phi^* + \tilde{U}(s)),\ (\mu + 1)^2\theta_{b/(\mu+1)} + s(\chi + \tilde{V}(s)),\ a(s)) \ : \ 0 < s \le \delta\}$$

for some $\chi \in X$ and $\delta > 0$. Here $(\tilde{U}(s), \tilde{V}(s), a(s))$ is a smooth function with respect to s and satisfies $(\tilde{U}(0), \tilde{V}(0), a(0)) = (0, 0, a^*)$ and $\int_\Omega \tilde{U}(s)\phi^* = 0$.

Proof. For $a > \lambda_1$, put $f(u,v) = u(a-u-cv)$ and $g(u,v) = v(b+du-v)$. Here, u and v are regarded as functions with respect to (U,V) (see (2.3)). By Taylor's expansion at the centre (U^*, V^*), we reduce differential equations of (EP) to the form

$$\begin{pmatrix} \Delta U \\ \Delta V \end{pmatrix} + \begin{pmatrix} f(u(U^*,V^*), v(U^*,V^*)) \\ g(u(U^*,V^*), v(U^*,V^*)) \end{pmatrix} + \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & g_v^* \end{pmatrix} \begin{pmatrix} u_U^* & u_V^* \\ v_U^* & v_V^* \end{pmatrix} \begin{pmatrix} U-U^* \\ V-V^* \end{pmatrix} + \begin{pmatrix} \rho^1(U-U^*, V-V^*) \\ \rho^2(U-U^*, V-V^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.9}$$

where $f_u^* := f_u(u(U^*,V^*), v(U^*,V^*))$, $u_U^* := u_U(U^*,V^*)$ and other notations are defined by similar rules. Here $\rho^i(U-U^*, V-V^*)\,(i = 1,2)$ are smooth functions such that $\rho^i(0,0) = \rho^i_{(U,V)}(0,0) = 0$. Since differentiation of (2.2) yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha v & \alpha u \\ -\dfrac{\beta v}{(1+\beta u)^2} & \mu + \dfrac{1}{1+\beta u} \end{pmatrix} \begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix},$$

some elementary calculations lead us to

$$\begin{pmatrix} u_U^* & u_V^* \\ v_U^* & v_V^* \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{\alpha(1+\beta\theta_a)\theta_a}{\mu(1+\beta\theta_a)+1} \\ 0 & \dfrac{1+\beta\theta_a}{\mu(1+\beta\theta_a)+1} \end{pmatrix}. \tag{2.10}$$

We note that $f(\theta_a, 0) = \theta_a(a-\theta_a) = -\Delta\theta_a$ and $g(\theta_a, 0) = 0$. So by virtue of (2.10), letting $(U^*, V^*) = (\theta_a, 0)$ and $\overline{U} := U - \theta_a$ in (2.9) implies

$$\begin{pmatrix} \Delta\overline{U} \\ \Delta V \end{pmatrix} + \begin{pmatrix} a-2\theta_a & -\dfrac{(\alpha a + c - 2\alpha\theta_a)(1+\beta\theta_a)\theta_a}{\mu(1+\beta\theta_a)+1} \\ 0 & \dfrac{(b+d\theta_a)(1+\beta\theta_a)}{\mu(1+\beta\theta_a)+1} \end{pmatrix} \begin{pmatrix} \overline{U} \\ V \end{pmatrix} + \begin{pmatrix} \rho^1(\overline{U}, V; a) \\ \rho^2(\overline{U}, V; a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.11}$$

where $\rho^i(\overline{U}, V; a)\,(i = 1,2)$ are smooth functions satisfying

$$\rho^1_{(\overline{U},V)}(0,0;a) = \rho^2_{(\overline{U},V)}(0,0;a) = 0 \quad \text{for all } a > \lambda_1. \tag{2.12}$$

Define a mapping $F : X \times \boldsymbol{R} \to Y$ by the left hand side of (2.11);

$$F(\overline{U}, V, a) = \begin{pmatrix} \Delta\overline{U} + (a-2\theta_a)\overline{U} - \dfrac{(\alpha a + c - 2\alpha\theta_a)(1+\beta\theta_a)\theta_a}{\mu(1+\beta\theta_a)+1} V + \rho^1(\overline{U}, V, a) \\ \Delta V + \dfrac{(b+d\theta_a)(1+\beta\theta_a)}{\mu(1+\beta\theta_a)+1} V + \rho^2(\overline{U}, V, a) \end{pmatrix}. \tag{2.13}$$

Since $(U, V) = (\theta_a, 0)$ is a semitrivial solution of (EP), it turns out $F(0, 0, a) = 0$ for $a > \lambda_1$. It follows from (2.12) and (2.13) that the Fréchet derivative of F at $(\overline{U}, V, a) = (0, 0, a)$ is given by

$$F_{(\overline{U},V)}(0,0,a)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + (a - 2\theta_a)h - \dfrac{(\alpha a + c - 2\alpha\theta_a)(1+\beta\theta_a)\theta_a}{\mu(1+\beta\theta_a)+1}k \\ \Delta k + \dfrac{(b + d\theta_a)(1+\beta\theta_a)}{\mu(1+\beta\theta_a)+1}k \end{pmatrix}.$$

By virtue of (2.6), we see that $\operatorname{Ker} F_{(\overline{U},V)}(0, 0, a)$ is nontrivial for $a = a_*$ and that

$$\operatorname{Ker} F_{(\overline{U},V)}(0, 0, a_*) = \operatorname{span}\{\psi, \phi_*\}.$$

Here ψ is defined by

$$\psi = -(-\Delta - a_* + 2\theta_{a_*})^{-1}\left\{\frac{(\alpha a_* + c - 2\alpha\theta_{a_*})(1+\beta\theta_{a_*})\theta_{a_*}}{\mu(1+\beta\theta_{a_*})+1}\phi_*\right\},$$

where $(-\Delta - a_* + 2\theta_{a_*})^{-1}$ is the inverse operator of $-\Delta - a_* + 2\theta_{a_*}$ with the homogeneous Dirichlet boundary condition on $\partial\Omega$. (Recall that $-\Delta - a_* + 2\theta_{a_*}$ is invertible; see, e.g., [5].) If $(\tilde{h}, \tilde{k}) \in \operatorname{Range} F_{(\overline{U},V)}(0, 0, a^*)$, then

$$\begin{cases} \Delta h + (a_* - 2\theta_{a_*})h - \dfrac{(\alpha a_* + c - 2\alpha\theta_{a_*})(1+\beta\theta_{a_*})\theta_{a_*}}{\mu(1+\beta\theta_{a_*})+1}k = \tilde{h} & \text{in } \Omega, \\ \Delta k + \dfrac{(b + d\theta_{a_*})(1+\beta\theta_{a_*})}{\mu(1+\beta\theta_{a_*})+1}k = \tilde{k} & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega \end{cases}$$

for some $(h, k) \in X$. It is well known that the second equation has a solution k if and only if $\int_\Omega \tilde{k}\phi_* = 0$. For such a solution k, the first equation has a unique solution h because $-\Delta - a_* + 2\theta_{a_*}$ is invertible. Then, it holds that $\operatorname{codimRange} F_{(\overline{U},V)}(0, 0, a^*) = 1$. In order to use the local bifurcation theory by Crandall-Rabinowitz [4] at $(\overline{U}, V, a) = (0, 0, a_*)$, we need to verify

$$F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix} \notin \operatorname{Range} F_{(\overline{U},V)}(0,0,a_*).$$

Since $\rho^i_{(\overline{U},V),a}(0, 0, a_*) = 0$ by (2.12), some elementary calculations from (2.13) enable us to obtain

$$F_{(\overline{U},V),a}(0,0,a_*)\begin{pmatrix} \psi \\ \phi_* \end{pmatrix}$$
$$= \begin{pmatrix} \left(1 - 2\left.\dfrac{\partial\theta_a}{\partial a}\right|_{a=a_*}\right)\psi - \dfrac{\partial}{\partial a}\left.\left\{\dfrac{(\alpha a + c - 2\alpha\theta_a)(1+\beta\theta_a)\theta_a}{\mu(1+\beta\theta_a)+1}\right\}\right|_{a=a_*}\phi_* \\ \left\{\dfrac{\mu d(1+\beta\theta_a)^2 + \beta(2d\theta_a + b) + d}{\{\mu(1+\beta\theta_a)+1\}^2}\right\}\left.\dfrac{\partial\theta_a}{\partial a}\right|_{a=a_*}\phi_* \end{pmatrix}.$$

Suppose for contradiction that there exists $k \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\Delta k + \frac{(b + d\theta_{a_*})(1 + \beta\theta_{a_*})}{\mu(1 + \beta\theta_{a_*}) + 1} k = \left\{ \frac{\mu d(1 + \beta\theta_a)^2 + \beta(2d\theta_a + b) + d}{\{\mu(1 + \beta\theta_a) + 1\}^2} \right\} \left. \frac{\partial \theta_a}{\partial a} \right|_{a=a_*} \phi_*.$$

Multiplying the above equation by ϕ_* and integrating the resulting expression, we have

$$\int_\Omega \left\{ \frac{\mu d(1 + \beta\theta_a)^2 + \beta(2d\theta_a + b) + d}{\{\mu(1 + \beta\theta_a) + 1\}^2} \right\} \left. \frac{\partial \theta_a}{\partial a} \right|_{a=a_*} \phi_*^2 = 0, \tag{2.14}$$

which is impossible. Because, the left hand side of (2.14) must be positive by the strict increasing property of θ_a with respect to a. Recall that $\overline{U} = U - \theta_a$, one can immediately obtain the assertion (i) by applying the local bifurcation theorem ([4]). We note that the possibility of other bifurcation points except $a = a_*$ is excluded by virtue of the Krein-Rutman theorem. In the case when $b > (\mu + 1)\lambda_1$, we can get the assertion (ii) by a similar bifurcation approach. □

2.4 Proof of Theorem 2.1

In this subsection, we will accomplish the proof of Theorem 2.1 by making use of the results in the previous subsections. First we will extend the local bifurcation branches Γ_1 and Γ_2 (obtained in Lemma 2.7) as global solution branches. By way of a result of these extensions, we obtain the following lemma.

Lemma 2.8. *If* $b < (\mu + 1)\lambda_1$ *and* $a > a_*$, *then* (EP) *possesses at least one positive solution. If* $b > (\mu + 1)\lambda_1$ *and* $a > a^*$, *then* (EP) *admits at least one positive solution.*

Proof. Let b satisfy $b < (\mu + 1)\lambda_1$. For the local bifurcation branch Γ_* obtained in Lemma 2.7, let $\hat{\Gamma}_*$ be a maximum extension of Γ_* in the direction $a > \lambda_1$ as a solution curve of (EP). According to the global bifurcation theory (Rabinowitz [27]), the following (i) or (ii) must hold true;

(i) $\hat{\Gamma}_*$ is unbounded in $X \times \boldsymbol{R}$;

(ii) $\hat{\Gamma}_*$ meets the trivial or a semitrivial solution curve at a certain point except for $(u, v, a) = (\theta_{a_*}, 0, a_*)$.

We introduce the following positive cone

$$P := \left\{ (u, v) \ : \ u > 0,\ v > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega \right\},$$

where ν is the unit outward normal to $\partial\Omega$. Assume that $(\hat{u}, \hat{v}, \hat{a}) \in \hat{\Gamma}_*$ satisfies $(\hat{u}, \hat{v}) \in \partial P$ and $\hat{a} > \lambda_1$. Then it follows that $\hat{u} \geq 0$, $\hat{v} \geq 0$ in $x \in \Omega$ and

$$\hat{u}(x_0)\hat{v}(x_0) = 0 \text{ at some } x_0 \in \Omega \tag{2.15}$$

or

$$\frac{\partial \hat{u}}{\partial \nu}(x_1)\frac{\partial \hat{v}}{\partial \nu}(x_1) = 0 \text{ at a certain } x_0 \in \partial\Omega. \tag{2.16}$$

By applying the strong maximum principle to (EP), it is possible to prove that each of (2.15) and (2.16) leads us to $\hat{u} \equiv 0$ or $\hat{v} \equiv 0$.

We now recall that positive solutions of (EP) bifurcate from the semitrivial solution curve $\{(\theta_a, 0, a) : a > \lambda_1\}$ and no positive solution bifurcates from the other semitrivial solution curve $\{(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a) : a > \lambda_1\}$. In addition, it is easily verified that the trivial solution is non-degenerate. Therefore, we deduce that $(\hat{u}, \hat{v}, \hat{a}) = (\theta_{a_*}, 0, a_*)$, which contradicts (ii). Thus (ii) is excluded and (i) must be satisfied. By taking account for the boundness for positive solutions to (EP) (Lemma 2.4) and the nonexistence result of positive solutions in the range $a < \lambda_1$, we can prove that Γ_* must be extended with respect to $a > \lambda_1$ as a positive solution curve of (EP). This global bifurcation property enables us to find at least one positive solution if $a > a_*$.

In the case when $b > (\mu + 1)\lambda_1$ and $a > a^*$, we can obtain the existence result of positive solutions to (EP) in a similar way. Thus the proof of Lemma 2.8 is complete.

□

By virtue of a one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$ in (2.2), Lemma 2.8 immediately implies Theorem 2.1.

3 A Large Nonlinear Diffusion Case

3.1 Two Shadow Systems as $\beta \nearrow \infty$

In what follows, we will concentrate ourselves on a special case when $\alpha = 0$ and β is sufficiently large. Our purpose is to derive the large nonlinear effect of β on the positive solution set of (SP). We will denote by $(\mathrm{SP})_0$ the problem (SP) with $\alpha = 0$:

$$(\mathrm{SP})_0 \begin{cases} \Delta u + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta\left[\left(\mu + \dfrac{1}{1 + \beta u}\right)v\right] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The following theorem assures existence of two shadow system as $\beta \to \infty$:

Theorem 3.1. *Suppose that $b > (\mu + 1)\lambda_1$. Let δ and ε be arbitrary small positive numbers. Then there exists a large number $B = B(\delta, \varepsilon)$ such that if*

$$a \in (\lambda_1, \lambda_1(c\mu\theta_{b/\mu}) - \delta\,] \cup [\,\lambda_1(c\mu\theta_{b/\mu}) + \delta, \delta^{-1}\,] \,(=: I_\delta)$$

and $\beta \geq B$, then any positive solution (u, v) of $(\mathrm{SP})_0$ *possesses either the next property* (i) *or* (ii) :

(i) *There exist a certain* $a_\infty \in I_\delta$ *and a certain positive solution* (u_∞, v_∞) *of*

$$\begin{cases} \Delta u_\infty + u_\infty(a_\infty - u_\infty - cv_\infty) = 0 & in\ \Omega, \\ \mu\Delta v_\infty + v_\infty(b + du_\infty - v_\infty) = 0 & in\ \Omega, \\ u_\infty = v_\infty = 0 & on\ \partial\Omega \end{cases} \tag{3.1}$$

such that $\|u - u_\infty\|_\infty + \|v - v_\infty\|_\infty + |a - a_\infty| < \varepsilon$.

(ii) *There exist a certain* $a_\infty \in I_\delta$ *and a certain positive solution* (w, v_∞) *of*

$$\begin{cases} \Delta w + w(a_\infty - cv_\infty) = 0 & in\ \Omega, \\ \Delta\left[\left(\mu + \dfrac{1}{1+w}\right)v_\infty\right] + v_\infty(b - v_\infty) = 0 & in\ \Omega, \\ w = v_\infty = 0 & on\ \partial\Omega \end{cases} \tag{3.2}$$

such that $\|\beta u - w\|_\infty + \|v - v_\infty\|_\infty + |a - a_\infty| < \varepsilon$.

Proof. We will accomplish the proof by a contradiction argument. Suppose that there exist a certain $\varepsilon_0 > 0$ and a sequence $\{(a_n, \beta_n)\} \subset I_\delta \times \boldsymbol{R}_+$ with $\lim_{n\to\infty} \beta_n = \infty$ such that all positive solutions (u_n, v_n) of $(\mathrm{SP})_0$ with $(a, \beta) = (a_n, \beta_n)$ satisfy

$$\|u_n - \tilde{u}\|_\infty + \|v_n - \tilde{v}\|_\infty + |a_n - \tilde{a}| \geq \varepsilon_0$$

for any positive solution $(\tilde{u}, \tilde{v}, \tilde{a})$ of (3.1) and

$$\|\beta_n u_n - w\|_\infty + \|v_n - \overline{v}\|_\infty + |a_n - \overline{a}| \geq \varepsilon_0$$

for any positive solution $(\overline{u}, \overline{v}, \overline{a})$ of (3.2).

If $\limsup_{n\to\infty} \beta_n \|u_n\|_\infty = \infty$, we can choose a subsequence with $\lim_{n\to\infty} \beta_n\|u_n\|_\infty = \infty$. For simplicity, we rewrite $\{(u_n, \beta_n)\}$ by such a subsequence. We now remember that Lemma 2.4 gives the following apriori estimates;

$$0 \leq u_n(x) \leq a_n \leq \frac{1}{\delta}, \quad 0 \leq V_n(x) \leq v_n(x) \leq (b + dM) + \frac{b}{\mu} \tag{3.3}$$

for all $x \in \Omega$ and $n \in \boldsymbol{N}$. Here we put $V_n := (\mu + \frac{1}{1+\beta_n u_n})v_n$. It follows from (3.3) and the first equation of $(\mathrm{SP})_0$ that for each $p > 1$ and $n \in \boldsymbol{N}$, $\|\Delta u_n\|_p \leq C$ with some constant C independent of n. Therefore, the standard elliptic regularity theory ([11]) enables us to obtain

$$\|u_n\|_{W^{2,p}} \leq C_2(\|u_n\|_p + \|\Delta u_n\|_p) \leq C_3$$

for some constants C_2 and C_3 independent of n. With the aid of the Ascoli-Arzelà's theorem, we can find a certain $u_\infty \in C^1(\overline{\Omega})$ with

$$\lim_{n\to\infty} u_n = u_\infty \ \text{ in } \ C^1(\overline{\Omega}) \tag{3.4}$$

subject to a suitable subsequence. In view of (3.3) and the second equation of $(\mathrm{SP})_0$, we can also find $V_\infty \in C^1(\overline{\Omega})$ such that

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty}\left(\mu + \frac{1}{1+\beta_n u_n}\right) v_n = V_\infty \ \text{ in } \ C^1(\overline{\Omega}) \tag{3.5}$$

by way of a subsequence. Next we will verify that

$$\lim_{n\to\infty} \frac{1}{1+\beta_n u_n} = \lim_{n\to\infty} \frac{1}{1+\beta_n \|u_n\|_\infty \tilde{u}_n} = 0 \ \text{ in } \ C^1(\overline{\Omega}). \tag{3.6}$$

Since $\tilde{u}_n := u_n/\|u_n\|_\infty$ satisfies

$$\Delta \tilde{u}_n + \tilde{u}_n(a_n - u_n - c v_n) = 0 \ \text{ in } \ \Omega, \ \ \tilde{u}_n|_{\partial\Omega} = 0, \tag{3.7}$$

then (3.3) and the elliptic regularity theory yield $\tilde{u}_\infty \in C^1(\overline{\Omega})$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}_\infty$ in $C^1(\overline{\Omega})$. By virtue of $\|\tilde{u}_\infty\|_\infty = 1$, we see $\tilde{u}_\infty > 0$ in Ω by the strong maximum principle. Hence $\lim_{n\to\infty} \beta_n \|u_n\|_\infty = \infty$ implies (3.6). Furthermore (3.3) gives some $v_\infty \in L^2(\Omega)$ such that

$$\lim_{n\to\infty} v_n = v_\infty \ \text{ weakly in } \ L^2(\Omega). \tag{3.8}$$

From (3.5), (3.6) and (3.8), we know that $v_\infty \in C^1(\overline{\Omega})$, and moreover that

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty}\left(\mu + \frac{1}{1+\beta_n u_n}\right) v_n = \mu v_\infty \ \text{ in } \ C^1(\overline{\Omega}). \tag{3.9}$$

It follows from (3.4) and (3.9) that (u_∞, v_∞) satisfies (3.1) with a certain $a_\infty \in \overline{I}_\delta$. In order to derive a contradiction, we will verify that both of u_∞ and v_∞ are positive functions in Ω. It follows from Lemma 2.5 and $\lim_{n\to\infty} v_n = v_\infty$ in $C^1(\overline{\Omega})$ that

$$v_\infty \geq \frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)} > 0. \tag{3.10}$$

Suppose for contradiction that $u_\infty \equiv 0$. Since v_∞ satisfies

$$\mu \Delta v_\infty + v_\infty(b - v_\infty) = 0 \ \text{ in } \ \Omega, \ \ v_\infty|_{\partial\Omega} = 0,$$

together with (3.10), we obtain $v_\infty = \mu\theta_{b/\mu}$. Letting $n \to \infty$ in (3.7) implies

$$\Delta \tilde{u}_\infty + \tilde{u}_\infty(a_\infty - c\mu\theta_{b/\mu}) = 0 \ \text{ in } \ \Omega, \ \ \tilde{u}_\infty|_{\partial\Omega} = 0.$$

Since $\tilde{u}_\infty > 0$ by the strong maximum principle, we know $a_\infty = \lambda_1(c\mu\theta_{b/\mu})$, which contradicts to $a_\infty \in \overline{I}_\delta$. So we must deduce that (u_∞, v_∞) is a positive solution of (3.1).

This property of (u_∞, v_∞) gives a contradiction for our assumption. So we accomplish the proof in a case when $\limsup_{n\to\infty} \beta_n \|u_n\|_\infty = \infty$.

If $\lim_{n\to\infty} \beta_n \|u_n\|_\infty < \infty$, then $w_n := \beta_n \|u_n\|_\infty$ are uniformly bounded with respect to n. By multiplying β_n by the first equation of $(\mathrm{SP})_0$, we obtain

$$\Delta w_n + w_n(a_n - u_n - cv_n) = 0 \ \text{ in } \ \Omega, \ \ w_n|_{\partial\Omega} = 0.$$

With use of (3.3) and the elliptic regularity, we can find a certain $w \in C^1(\overline{\Omega})$ such that

$$\lim_{n\to\infty} w_n = w \ \text{ in } \ C^1(\overline{\Omega}), \tag{3.11}$$

subject to a subsequence. Hence (3.11) implies

$$\lim_{n\to\infty} \frac{1}{1 + w_n} = \frac{1}{1+w} \ \text{ in } \ C^1(\overline{\Omega}). \tag{3.12}$$

Along a similar argument to the previous case, we obtain $V_\infty \in C^1(\overline{\Omega})$ with

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \left(\mu + \frac{1}{1 + w_n}\right) v_n = V_\infty \ \text{ in } \ C^1(\overline{\Omega}). \tag{3.13}$$

Together with the L^2 weak compactness property of $\{v_n\}$ (see (3.8)), (3.12) and (3.13) yield $V_\infty = (\mu + \frac{1}{1+w}) v_\infty$. Therefore by letting $n \to \infty$ in $(\mathrm{SP})_0$ with $(u, v, a, \beta) = (u_n, v_n, a_n, \beta_n)$, we see that (w, v_∞) satisfies (3.2). Furthermore we can also prove that (w, v_∞) is a positive solution to (3.2) by a similar argument to the previous case (see [15] for details). However this conclusion contradicts our assumption. So we complete the proof of Theorem 3.1. □

3.2 First Shadow System (3.1)

In this subsection, we introduce the positive solution set to the first shadow system (3.1), which has been discussed by many mathematicians (e.g., [2], [5], [6], [7], [17], [18], [19], [20], [31]). As a summary of their all results, we know the next result about the positive solution set of (3.1).

Theorem 3.2. *Let $\hat{a} = \lambda_1(c\mu\theta_{b/\mu})$. If $b > \mu\lambda_1$, then* (3.1) *has a positive solution if and only if $a > \hat{a}$. From the bifurcation structure point of view, the positive solution set of* (3.1). *contains a local bifurcation branch $\Gamma_1 = \{(u(s), v(s), a(s)) \in X \times \boldsymbol{R} : s \in (0, \delta)\}$, such that $(u(0), v(0), a(0)) = (0, \mu\theta_{b/\mu}, \hat{a})$. Furthermore, Γ_1 can be extended in the direction $a > \hat{a}$ as an unbounded positive solution branch of* (3.1). *In a special case when $N = 1$, uniqueness of positive solutions holds true.*

3.3 Second Shadow System (3.2)

In this subsection, we discuss the second shadow system (3.2). Letting

$$V(x) := \left(\mu + \frac{1}{1 + w(x)}\right) v(x) \tag{3.14}$$

in (3.2), we obtain the following semilinear elliptic system;

$$\begin{cases} \Delta w + w\left\{a - \dfrac{c(1+w)}{\mu(1+w)+1}V\right\} = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{1+w}{\mu(1+w)+1}V\left\{b - \dfrac{1+w}{\mu(1+w)+1}V\right\} = 0 & \text{in } \Omega, \\ w = V = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

We will concentrate ourselves on (3.15), because we discuss nonnegative solutions. The following lemma gives apriori bounds of v and V.

Lemma 3.3. *Let* (w, v) *be any positive solution of* (3.2) *and let* (w, V) *be any positive solution of* (3.15). *Then for all* $x \in \Omega$,

$$\frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)}(x) < v(x) < \frac{(\mu+1)^2}{\mu}\theta_{b/\mu}(x), \quad \text{and} \quad \mu^2\theta_{b/(\mu+1)}(x) < V(x) < (\mu+1)^2\theta_{b/\mu}(x).$$

This lemma can be proved by a standard comparison argument. We refer to [15] for the proof. With the aid of Lemma 3.3, we obtain the next nonexistence region of positive solutions to the second shadow system.

Lemma 3.4. *Suppose that* $b > (\mu+1)\lambda_1$. *If*

$$a \le \lambda_1\left(\frac{c\mu^2}{\mu+1}\theta_{b/(\mu+1)}\right) \quad \text{or} \quad a \ge \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right),$$

then both of (3.2) *and* (3.15) *have no positive solution.*

Proof. From the first equation of (3.2), we see

$$-\Delta w + cvw = aw \text{ in } \Omega, \quad w|_{\partial\Omega} = 0. \tag{3.16}$$

Note that w is a positive solution of (3.16) if and only if $a = \lambda_1(cv)$. By taking account for the monotone increasing property of $\lambda_1(q)$ with respect to $q \in C(\overline{\Omega})$, we get from Lemma 3.3

$$\lambda_1\left(\frac{c\mu^2}{\mu+1}\theta_{b/(\mu+1)}\right) < a = \lambda_1(cv) < \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right),$$

provided w is a positive solution of (3.16), So we complete the proof of Lemma 3.4. □

In the case when $\alpha = 0$, the positive number a^* defined in (2.7) can be expressed as

$$a^* := \lambda_1(c(\mu + 1)\theta_{b/(\mu+1)}). \tag{3.17}$$

In this case, the associate positive eigenfunction ϕ^* (see (2.8)) satisfies

$$-\Delta\phi^* + \{c(\mu + 1)\theta_{b/(\mu+1)} - a^*\}\phi^* = 0 \text{ in } \Omega, \quad \phi^* = 0 \text{ on } \partial\Omega, \quad \|\phi^*\| = 1.$$

Hence (3.15) has a semitrivial solution $(w, V) = (0, (\mu + 1)^2\theta_{b/(\mu+1)})$. Positive solutions of (3.15) bifurcate from the semitrivial solution curve $(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a^*) \in X \times \boldsymbol{R}$ at the same point $a = a^*$ to the original (EP) case:

Lemma 3.5. *Suppose that $b > (\mu + 1)\lambda_1$. Positive solutions of* (3.15) *bifurcate from the semitrivial solution curve $\{(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a) : a > \lambda_1\}$ if and only if $a = a^*$. To be precise, all positive solutions of* (3.15) *near* $(0, (\mu + 1)^2\theta_{b/(\mu+1)}, a^*) \in X \times \boldsymbol{R}$ *can be parameterized as*

$$\Gamma_\delta := \{(s(\phi^* + \tilde{W}(s)), (\mu + 1)^2\theta_{b/(\mu+1)} + s(\chi + \tilde{V}(s)), a(s)) : 0 < s \le \delta\}$$

for some $\delta > 0$ and $\chi \in X$. Here $(\tilde{W}(s), \tilde{V}(s), a(s))$ is a smooth function with respect to s and satisfies $(\tilde{W}(0), \tilde{V}(0), a(0)) = (0, 0, a^*)$ and $\int_\Omega \tilde{W}(s)\phi^* = 0$.

Lemma 3.5 can be proved along a similar bifurcation argument to the proof of Lemma 2.7 (see [15]). Here we should note that

$$a^* < \hat{a}\,(:= \lambda_1(c\mu\theta_{b/\mu})), \tag{3.18}$$

if $b > (\mu + 1)\lambda_1$. We refer to [15] for the proof of (3.18).

Lemma 3.6. *Let Γ_δ be the local bifurcation branch obtained in Lemma 3.5. If $b > (\mu + 1)\lambda_1$, then Γ_δ ($\subset X \times \boldsymbol{R}$) can be extended as an unbounded positive solution branch Γ of* (3.15). *Furthermore, Γ contains a parametrized subset*

$$\{(w(s), V(s), a(s)) \in X \times \boldsymbol{R} : s \in (C, \infty)\}, \tag{3.19}$$

such that $\lim_{s\to\infty} \|w(s)\|_\infty = \infty$, $\lim_{s\to\infty} V(s) = \mu^2\theta_{b/\mu}$ in $C^1(\overline{\Omega})$ and $\lim_{s\to\infty} a(s) = \hat{a}$, where $\hat{a}$ is the positive number defined in (3.18).

Proof. Along a global bifurcation argument as the proof of Lemma 2.8, we can extend Γ_δ as an unbounded positive solution branch Γ of (3.15). By virtue of apriori bounds for v and a (Lemmas 3.3 and 3.4), we must deduce that Γ_δ is unbounded with respect to $\|w\|_{W^{1,p}}$. Then there exists a positive solution sequence $\{(w_n, V_n, a_n)\} \subset \Gamma$ such that $\lim_{n\to\infty} \|w_n\|_{W^{1,p}} = \infty$. By the first equation of (3.15), we know $\lim_{n\to\infty} \|w_n\|_\infty = \infty$. Since $\{a_n\}$ is a bounded sequence by Lemma 3.4, we can put $a_\infty := \lim_{n\to\infty} a_n$, subject to a subsequence. Furthermore let $\tilde{w}_n := w_n/\|w_n\|_\infty$. So a compactness argument as the proof of Theorem 3.1 enables us to find a certain $(\tilde{w}, v_\infty) \in C^1(\overline{\Omega})^2$ such that

$$\lim_{n\to\infty}(\tilde{w}_n, V_n) = (\tilde{w}, \mu v_\infty) \text{ in } C^1(\overline{\Omega})^2,$$

and moreover,

$$\begin{cases} \Delta\tilde{w} + \tilde{w}(a_\infty - cv_\infty) = 0 & \text{in } \Omega, \\ \mu\Delta v_\infty + v_\infty(b - v_\infty) = 0 & \text{in } \Omega, \\ \tilde{w} = \tilde{v}_\infty = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

by way of a subsequence. Since

$$\frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)} \le v_\infty \le \frac{(\mu+1)^2}{\mu}\theta_{b/\mu} \text{ in } \Omega$$

by Lemma 3.3, the second equation of (3.20) implies $v_\infty = \mu\theta_{b/\mu}$. Therefore, we obtain $a_\infty = \lambda_1(c\mu\theta_{b/\mu})$ by the first equation of (3.20). We refer to [15] for the proof of the expression (3.19).

□

By the one-to-one correspondence between $(w, v) > 0$ and $(w, V) > 0$ ((3.14), we obtain such information on the positive solution set of (3.2), as a summary of Lemmas 3.3-3.6:

Theorem 3.7. *If $b > (\mu+1)\lambda_1$, then the positive solution set of* (3.2) *contains a local bifurcation branch $\Gamma_2 = \{(w(s), v(s), a(s)) \in X \times \mathbf{R} \ : \ s \in (0, \delta)\}$, such that $(w(0), v(0), a(0)) = (0, (\mu+1)\theta_{b/(\mu+1)}, a^*)$. Furthermore, Γ_2 can be extended as an unbounded positive solution branch $\hat{\Gamma}_2$ of* (3.2) *with the following properties*:

(i) *Any $(w, v, a) \in \hat{\Gamma}_2$ satisfies*

$$\frac{\mu^2}{\mu+1}\theta_{b/(\mu+1)} < v < \frac{(\mu+1)^2}{\mu}\theta_{b/\mu} \ \textit{and}\ \lambda_1\left(\frac{c\mu^2}{\mu+1}\theta_{b/(\mu+1)}\right) < a < \lambda_1\left(\frac{c(\mu+1)^2}{\mu}\theta_{b/\mu}\right).$$

(ii) *$\hat{\Gamma}_2$ contains a parametrized subset $\{(w(s), v(s), a(s)) \in X \times \mathbf{R} \ : \ s \in (C, \infty)\}$, such that $\lim_{s\to\infty}\|w(s)\|_\infty = \infty$, $\lim_{s\to\infty} v(s) = \mu\theta_{b/\mu}$ in $C^1(\overline{\Omega})$ and $\lim_{s\to\infty} a(s) = \hat{a}$.*

3.4 Convergence to Limiting Solutions as $\beta \nearrow \infty$

By a combination of Theorems 3.1, 3.2 and 3.7, we can obtain the next convergence properties of positive solutions of the original system $(\mathrm{SP})_0$ as $\beta \to \infty$. We refer to [15] for the proof.

Theorem 3.8. *Suppose $b > (\mu + 1)\lambda_1$. Let $\{(u_n, v_n)\}$ be any positive solution sequence of* (SP)$_0$ *with $\beta = \beta_n$ and $\lim_{n\to\infty} \beta_n = \infty$. Then the following convergence properties* (i) *and* (ii) *hold true:*

(i) *If $a \in (\hat{a}, \infty)$, $\lim_{n\to\infty}(u_n, v_n) = (u, v)$ in $L^\infty(\Omega)^2$ (subject to a subsequence) with some positive solution (u, v) of* (3.1).

(ii) *If $a \in (a^*, \hat{a})$, $\lim_{n\to\infty}(\beta_n u_n, v_n) = (w, v)$ in $L^\infty(\Omega)^2$ (subject to a subsequence) with some positive solution (w, v) of* (3.2). *In this case, $\|u_n\|_\infty = O(1/\beta_n)$ for sufficiently large n.*

In the sense of the above theorem, we can say that the positive solution set of (SP)$_0$ changes near $a = \hat{a}$ structurally, if β is sufficiently large. We should remark that if $a \in (a^*, \hat{a})$, *any* positive solution (u, v) of (SP)$_0$ must satisfy $\|u\|_\infty = O(1/\beta)$ when β is large enough, because the first shadow system (3.1) has no positive solution if $a < \hat{a}$.

References

[1] H. Amann, *Dynamic theory of quasilinear parabolic equations-III. Global existence*, Math. Z., **202** (1989), 219–250.

[2] J. Blat and K. J. Brown, *Bifurcation of steady-state solutions in predator-prey and competition systems*, Proc. Royal. Soc. Edinburgh, **97A** (1984), 21–34.

[3] Y. S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion*, Discrete Contin. Dynam. Systems, **10** (2004), 719–730.

[4] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal., **8** (1971), 321–340.

[5] E. N. Dancer, *On positive solutions of some pairs of differential equations*, Trans. Amer. Math. Soc., **284** (1984), 729–743.

[6] E. N. Dancer, *On positive solutions of some pairs of differential equations, II*, J. Differential Equations, **60** (1985), 236–258.

[7] E. N. Dancer, *On uniqueness and stability for solutions of singularly perturbed predator-prey type equations with diffusion*, J. Differential Equations, **102** (1993), 1–32.

[8] L. Dung, *Cross diffusion systems on n spatial dimension domains*, Indiana Univ. Math. J., **51** (2002), 625–643.

[9] L. Dung, *Global existence for a class of strongly coupled parabolic systems*, to appear in Annli. Mat..

[10] L. Dung, L. V. Nguyen and T. T. Nguyen, *Shigesada-Kawasaki-Teramoto model on higher dimensional domains*, Electronic J. Differential Equations, **72** (2003), 1–12.

[11] D. Gilbarg and N. S. Trudinger, "Elliptic partial differential equations of second order. Second edition", Springer-Verlag, Berlin, 1983.

[12] T. Kadota and K. Kuto, *Positive steady-states to a prey-predator model with some nonlinear diffusion effects*, preprint.

[13] Y. Kan-on, *Stability of singularly perturbed solutions to nonlinear diffusion systems arising in population dynamics*, Hiroshima Math. J., **23** (1993), 509–536.

[14] K. Kuto, *Stability of steady-state solutions to a prey-predator system with cross-diffusion*, J. Differential Equations, **197** (2004), 293–314.

[15] K. Kuto, *Large nonlinear diffusion effects on the steady-state solution set of a prey-predator population model*, preprint.

[16] K. Kuto and Y. Yamada, *Multiple coexistence states for a prey-predator system with cross-diffusion*, J. Differential Equations, **197** (2004), 315–348.

[17] L. Li, *Coexistence theorems of steady states for predator-prey interacting system*, Trans. Amer. Math. Soc., **305** (1988), 143–166.

[18] L. Li, *On positive solutions of a nonlinear equilibrium boundary value problem*, J. Math. Anal. Appl., **138** (1989), 537–549.

[19] J. López-Gómez, R. Pardo, *Coexistence regions in Lotka-Volterra models with diffusion*, Nonlinear Anal. TMA., **19** (1992), 11–28.

[20] J. López-Gómez and R. Pardo, *Existence and uniqueness of coexistence states for the predator-prey model with diffusion*, Differential Integral Equations, **6** (1993), 1025–1031.

[21] Y. Lou and W.-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, **131** (1996), 79–131.

[22] Y. Lou and W.-M. Ni, *Diffusion vs cross-diffusion : An elliptic approach*, J. Differential Equations, **154** (1999), 157–190.

[23] Y. Lou, W.-M. Ni and S. Yotsutani, *On a limiting system in the Lotka-Volterra competition with cross-diffusion*, Discrete Contin. Dynam. Systems, **10** (2004), 435-458.

[24] Y. Lou, W.-M. Ni and Y. Wu, *On the global existence of a cross-diffusion system*, Discrete Contin. Dynam. Systems, **4** (1998), 193–203.

[25] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, *Coexistence problem for two competing species models with density-dependent diffusion*, Hiroshima Math. J., **14** (1984), 425–449.

[26] K. Nakashima and Y. Yamada, *Positive steady states for prey-predator models with cross-diffusion*, Adv. Differential Equations, **6** (1996), 1099-1122.

[27] R. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., **7** (1971), 487–513.

[28] K. Ryu and I. Ahn, *Positive steady-states for two interacting species models with linear self-cross diffusions*, Discrete Contin. Dynam. Systems, **9** (2003), 1049–1061.

[29] N. Shigesada, K.Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theor. Biol., **79** (1979), 83–99.

[30] A. Yagi, *Global solution to some quasilinear parabolic system in population dynamics*, Nonlinear Anal. TMA., **21** (1993), 531–556.

[31] Y. Yamada, *Stability of steady states for prey-predator diffusion equations with homogeneous Dirichlet conditions*, SIAM J. Math. Anal., **21** (1990), 327–345.